Course 212: Academic Year 1998–9
Problems I

1. Let $I_1, I_2, I_3, I_4, I_5, \ldots$ be an infinite sequence of closed intervals in $\mathbb{R}$, where each interval I_n is given by

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$$

for some real numbers a_n and b_n satisiying $a_n \leq b_n$. Suppose that $I_{n+1} \subset I_n$ for each natural number n and that $b_n - a_n \to 0$ as $n \to +\infty$. Prove that there exists exactly one real number c with the property that c belongs to I_n for each natural number n. [Hint: use the theorem on the convergence of bounded monotonic sequences.]

2. Determine which of the following functions are homeomorphisms:—

   (a) the function $c: \mathbb{R} \to \mathbb{R}$ defined by $c(t) = t^3$;

   (b) the function $g: H \to H$ defined by $g(x, y) = (x, y^2)$, where $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$;

   (c) the function $h: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x, y) = (x, y^2)$;

   (d) the function $p: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $p(x, y) = (x, x^2 + 3y)$;

   (e) the function $q: D_1 \to D_2$ defined by $q(x, y) = (e^{-y} \cos x, e^{-y} \sin x)$, where

$$\begin{aligned} D_1 &= \{(x, y) \in \mathbb{R}^2 : y > 0 \text{ and } 0 < x < \pi\}, \\ D_2 &= \{(x, y) \in \mathbb{R}^2 : y > 0 \text{ and } x^2 + y^2 < 1\}. \end{aligned}$$

3. Consider the following subsets of $\mathbb{R}^3$. Determine which are open and which are closed in $\mathbb{R}^3$. [Fully justify your answers.]

   (a) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \geq 3\}$,

   (b) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z < 1\}$,

   (c) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \geq 7 \text{ and } z \leq 0\}$,

   (d) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 7 \text{ or } z > 2\}$,

   (e) $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } x^2 - y^2 - z^2 = 1\}$,

   (f) $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } y^2 + z^2 = 1/x\}$.

4. Let $D = \{(x, y) \in \mathbb{R}^2 : y < f(x)\}$, where $f: \mathbb{R} \to \mathbb{R}$ is continuous. Prove that D is open in $\mathbb{R}^2$.

5. Let X be a subset of $\mathbb{R}^n$, let $\mathbf{x}$ be a point of X, and let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that does not belong to X. Let S be the set of all non-negative real numbers t with the property that $(1-\tau)\mathbf{x} + \tau\mathbf{p} \in X$ for all $\tau \in [0, t]$ (i.e., the line segment joining the point $\mathbf{x}$ to the point $(1-t)\mathbf{x} + t\mathbf{p}$ is contained within the set X). Let $s = \sup S$, and let $\mathbf{y} = (1-s)\mathbf{x} + s\mathbf{p}$.

   (a) Explain why $0 \leq s \leq 1$.

   (b) Show that if the set X is closed in $\mathbb{R}^n$ then $\mathbf{y} \in X$.

   (c) Show that if the set X is open in $\mathbb{R}^n$ then $\mathbf{y} \in \mathbb{R}^n \setminus X$.

   (d) Using (a) and (b), show that the only subsets of $\mathbb{R}^n$ that are both open and closed in $\mathbb{R}^n$ are the empty set $\emptyset$ and $\mathbb{R}^n$ itself.

6. Let $d_1: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $d_2: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be be the functions defined by

$$\begin{aligned} d_1((x,y),(u,v)) &= |x-u| + |y-v|, \\ d_2((x,y),(u,v)) &= \max(|x-u|, |y-v|) \end{aligned}$$

   for all real numbers x, y, u and v. Verify that the metric space axioms are satisfied by the distance functions d_1 and d_2 on $\mathbb{R}^2$. What are the shapes of the open balls in $\mathbb{R}^2$ defined using these distance functions.

7. Let X be a non-empty set. Let $d: X \times X \to \mathbb{R}$ be defined so that $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. Verify that the distance function d on X satisfies the metric space axioms.

8. Let X be a metric space with distance function d, and let A be a non-empty subset of X. Let $f: X \to \mathbb{R}$ be the function defined by $f(x) = \inf\{d(x,a) : a \in A\}$ (i.e., $f(x)$ is the largest real number with the property that $f(x) \leq d(x,a)$ for all $a \in A$). Use the Triangle Inequality to prove that $f(x) \leq f(y) + d(x,y)$ for all $x, y \in X$, and hence show that $|f(x) - f(y)| < d(x,y)$. (Note that this implies that the function $f: X \to \mathbb{R}$ is continuous.) Prove that A is closed in X if and only if $A = \{x \in X : f(x) = 0\}$.

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Problems II

1. The *Zariski topology* on the real numbers $\mathbb{R}$ is the topology whose open sets are the empty set, the set $\mathbb{R}$ itself and those subsets of $\mathbb{R}$ whose complements are finite. [**Note added in 2016:** this topology is also known as the *cofinite topology* on $\mathbb{R}$.]

   (a) Prove that any polynomial function from $\mathbb{R}$ to itself is continuous with respect to the Zariski topology in $\mathbb{R}$.

   (b) Give an example of a function from $\mathbb{R}$ to itself which is continuous with respect to the usual topology on $\mathbb{R}$ but is not continuous with respect to the Zariski topology on $\mathbb{R}$.

2. (a) Let $f\colon X \to Y$ be a function from a topological space X to a topological space Y, and let A and B be subsets of X for which $X = A \cup B$. Suppose that the restrictions $f|A$ and $f|B$ of f to the sets A and B are continuous. Is $f\colon X \to Y$ necessarily continuous on X? [Give proof or counterexample.]

   (b) Let $f\colon X \to Y$ be a function from a topological space X to a topological space Y, and let $\mathcal{F}$ be a (not necessarily finite) collection of closed subsets of X whose union is the whole of X. Suppose that the restriction $f|A$ of f to A is continuous for all closed sets A in the collection $\mathcal{F}$. Is $f\colon X \to Y$ necessarily continuous on X? [Give proof or counterexample.]

3. Let $f\colon X \to Y$ be a function from a topological space X to a topological space Y, and let $\mathcal{U}$ be a collection of open subsets of X whose union is the whole of X. Suppose that the restriction $f|W$ of f to W is continuous for all open sets W in the collection $\mathcal{U}$. Prove that $f\colon X \to Y$ is continuous on X.

4. Let X be a topological space, let A be a subset of X, and let B be the complement $X \setminus A$ of A in X. Prove that the interior of B is the complement of the closure of A.

5. (a) Let X and Y be metric spaces, and let $d_1\colon X \times Y \to \mathbb{R}$ and $d_y\colon X \times Y \to \mathbb{R}$ be the functions defined by

$$\begin{aligned} d_1((x,y),(u,v)) &= d(x,u) + d(y,v), \\ d_2((x,y),(u,v)) &= \max(d(x,u), d(y,v)) \end{aligned}$$

for all real numbers x, y, u and v. Verify that the metric space axioms are satisfied by the distance functions d_1 and d_2 on $X \times Y$.

(b) Show that the topology on $X \times Y$ generated by the distance functions d_1 and d_2 defined in (a) is the product topology on $X \times Y$.

6. Let S^1 be the unit circle in $\mathbb{R}^2$. Prove that the map $r\colon [0,1] \times [0,1] \to S^1 \times [0,1]$ that sends (t, τ) to $((\cos 2\pi t, \sin 2\pi t), \tau)$ for all $t \in [0,1]$ and $\tau \in [0,1]$ is an identification map.

7. Determine which of the following subsets of $\mathbb{R}^3$ are compact:—

   (i) the x-axis $\{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$;

   (ii) the surface of a tetrahedron in $\mathbb{R}^3$;

   (iii) $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } x^2 + y^2 - z^2 \leq 1\}$.

8. Let X be a topological space. Suppose that $X = A \cup B$, where A and B are path-connected subsets of X and $A \cap B$ is non-empty. Show that X is path-connected.

9. Let $f\colon X \to Y$ be a continuous map between topological spaces X and Y. Suppose that X is path-connected. Prove that the image $f(X)$ of the map f is also path-connected.

10. Let X and Y be path-connected topological spaces. Explain why the Cartesian product $X \times Y$ of X and Y is path-connected.

11. Determine the connected components of the following subsets of $\mathbb{R}^2$:

    (i) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,

    (ii) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$,

    (iii) $\{(x, y) \in \mathbb{R}^2 : y^2 = x(x^2 - 1)\}$,

    (iv) $\{(x, y) \in \mathbb{R}^2 : (x - n)^2 + y^2 > \frac{1}{4} \text{ for all } n \in \mathbb{Z}\}$.

    [Fully justify your answers.]

12. A topological space X is said to be *locally path-connected* if, given any point x of X there exists a path-connected open set U in X such that $x \in U$.

    (a) Let X be a locally path-connected topological space, and let p be a point of X. Let A be the set of all points x of X for which there exists a path from p to x, and let B be the complement of A in X. Prove that A and B are open in X.

(b) Use the result of (a) to show that any connected and locally path-connected topological space is path-connected.

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Problems III

1. (a) Let V be the real vector space consisting of all continuous functions from $[-1,1]$ to $\mathbb{R}$, and let $\|f\|_1 = \int_{-1}^{1} |f(x)|\,dx$ for all $f \in V$. Prove that $\|.\|_1$ is a norm on V.

   (b) Prove that V, with the norm $\|.\|_1$ defined above, is not a Banach space. [Hint: consider the infinite sequence $f_1, f_2, f_3, \ldots$ in V, where

$$f_j(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq -1/j; \\ jx & \text{if } -1/j \leq x \leq 1/j; \\ 1 & \text{if } 1/j \leq x \leq 1.] \end{cases}$$

2. Let X be a Banach space, and let S be an invertible bounded linear operator on X (i.e., a bounded linear transformation from X to itself which is has a bounded inverse S^{-1}.) Prove that if T is a bounded linear operator on X satisfying $\|T - S\| < \|S^{-1}\|$ then T is invertible. (Note that this implies that the set of invertible operators is open in the space $B(X)$ of bounded linear operators on X.)

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Problems IV

1. What is the winding number about zero of the closed curve $\gamma_1\colon [0,1] \to \mathbb{C}$, where $\gamma_1(t) = 2\cos 6\pi t - e^{t^2-t}\sin 8\pi t + (3\sin 6\pi t + e^{t^2-t}\cos 8\pi t)i$?

2. Show that if a closed curve γ in the complex plane does not intersect the negative real axis $\{t \in \mathbb{R} : t \leq 0\}$ then $n(\gamma, 0) = 0$.

3. Let $\gamma_1\colon [0,1] \to \mathbb{C}$ and $\gamma_2\colon [0,1] \to \mathbb{C}$ be closed curves in the complex plane that do not pass through zero, and let $\eta\colon [0,1] \to \mathbb{C}$ be the closed curve given by $\eta(t) = \gamma_1(t)\gamma_2(t)$ for all $t \in [0,1]$. Prove that $n(\eta, 0) = n(\gamma_1, 0) + n(\gamma_2, 0)$.

4. Let $\gamma_1\colon [0,1] \to \mathbb{C}$ and $\gamma_2\colon [0,1] \to \mathbb{C}$ be closed curves in the complex plane satisfying $\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1)$ that do not pass through some complex number w. The *concatenation* $\gamma_1.\gamma_2\colon [0,1] \to \mathbb{C}$ of γ_1 and γ_2 is defined by the formula

$$(\gamma_1.\gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

   Prove that $n(\gamma_1.\gamma_2, w) = n(\gamma_1, w) + n(\gamma_2, w)$.

5. Let $\gamma_0\colon [0,1] \to \mathbb{C}$ and $\gamma_1\colon [0,1] \to \mathbb{C}$ be closed curves in the complex plane that do not pass through some complex number w. Suppose that $n(\gamma_0, w) = n(\gamma_1, w)$. Prove that there exists a continuous function $F\colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{w\}$ such that $F(t, 0) = \gamma_0(t)$ and $F(t, 1) = \gamma_1(t)$ for all $t \in [0,1]$ and $F(0, \tau) = F(1, \tau)$ for all $\tau \in [0,1]$. [Hint: define F in terms of lifts $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ of γ_0 and γ_1, where $\exp(\tilde{\gamma}_0(t)) = \gamma_0(t) - w$ and $\exp(\tilde{\gamma}_1(t)) = \gamma_1(t) - w$ for all $t \in [0,1]$.]

   (Note that for each $\tau \in [0,1]$, the function $t \mapsto F(t, \tau)$ is a closed curve that does not pass through w. The result of this question is thus the converse of the result that winding numbers are preserved under continuous deformations.)